ORDER σ-CONTINUOUS OPERATORS ON BANACH LATTICES

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The aim of this paper is to extend Lozanovskii's results on Banach lattices having order σ-continuous norms (see [8] for details) to operators defined on Banach lattices.

Let E be a Banach lattice and let F be a Banach space.

An operator $T \in L(E,F)$ is said to be of <u>type A</u> provided that

$0 \leq x_n \downarrow$ in E implies $(Tx_n)_n$ is norm convergent in F.

T is said to be of <u>type B</u> provided that

$0 \leq x_n \uparrow$, $\|x_n\| \leq K$ in E implies $(Tx_n)_n$ is norm convergent in F.

The identity of an order σ-complete Banach lattice E is an operator of type A (respectively of type B) iff E has order σ-continuous norm (respectively E is weakly sequentially complete).

Our main results are as follows

<u>**THEOREM A**</u>. Let E be an almost σ-complete Banach lattice (the relevant definition appears below), let F be a Banach space and let $T \in L(E,F)$. Then the following assertions are equivalent:

i) T is of type A ;

ii) T'' maps the ideal I_E (generated by E in E'') into F ;

iii) T has the Pelczynski's property (u), i.e. for each weak Cauchy sequence $(x_n)_n$ in E there is a weakly summable sequence $(y_n)_n$ in $\overline{T(E)}$ such that $Tx_n - \sum_{k=1}^{n} y_k \xrightarrow{w} 0$;

iv) There exists no subspace X of E, isomorphic to ℓ^∞, such that $T|X$ is an isomorphism.

THEOREM B. Let E be a Banach lattice, F a Banach space and $T \in L(E,F)$. Then T is of type B iff there exists no sublattice X of E, lattice isomorphic to c_0, such that $T|X$ is an isomorphism.

Related results are discussed in [12].

The author is much indebted to P.G.Dodds for providing him with a copy of [2].

1. PRELIMINARIES

The main ingredients which we need to characterize the operators of type A are a very general scheme to associate AM- and AL- spaces to a given Banach lattice and some consequences of Grothendieck's criterion of weak compactness in a space $C(S)'$.

Let E be a Banach lattice and let $x \in E$, $x \geq 0$. We consider the ideal E_x generated by x in E

$$E_x = \{ y \in E \; ; \; (\exists) \; \alpha \geq 0 \text{ such that } |y| \leq \alpha x \}$$

endowed with the norm

$$\|y\|_x = \inf \{ \alpha \; ; \; |y| \leq \alpha x \} .$$

Then E_x is an AM- space with a strong order unit (which is x) and thus order isometric to a space $C(S_x)$ for some compact (Hausdorff) S_x.

If $x'' \in E''$, $x'' \geq 0$, then the Banach lattice $E_{x''} = E''_{x''} \cap E$ endowed with the norm induced by $\| \; \|_{x''}$ is also an AM- space and the canonical inclusion $i_{x''}: E_{x''} \longrightarrow E$ is an interval preserving mapping.

For each $x' \in E'$, $x' \geq 0$, we consider on E the following relation of equivalence

$$x \sim y \quad \text{iff} \quad x'(|x-y|) = 0 .$$

The completion of $E/\sim$ with respect to the norm

$$\|x\|_{x'} = x'(|x|)$$

is an AL- space, denoted by $L^1(x')$. Let us denote by $j_{x'}: E \longrightarrow L^1(x')$ the canonical surjection. Then $(j_{x'})' = i_{x'}$.

The prerequisites which we need on weakly compact operators defined on $C(S)$- spaces are essentially contained in the following

1.1 THEOREM .Let S be a compact Hausdorff space,E a Banach space and $T \in L(C(S),E)$.Then the following assertions are equivalent:

i) T is weakly compact ;

ii) T maps every bounded sequence of pairwise disjoint elements into a norm convergent sequence ;

iii) T maps every monotone bounded sequence into a norm convergent sequence ;

iv) There exists no sublattice X of $C(S)$,order isomorphic to c_o , such that $T|X$ is an isomorphism ;

v) There exists a positive Radon measure μ on S such that T is absolutely continuous with respect to μ , i.e.

$$\|T(\cdot)\| \leq \varepsilon\|\cdot\| + \delta(\varepsilon)\cdot\mu(|\cdot|)$$

for each $\varepsilon > 0$;

vi) T maps every bounded sequence into a sequence with stable sub - sequences .

Recall that a sequence $(x_n)_n$ of elements of E is said to be stable (with limit x) if there exists an $x \in E$ such that $\lim_{n\to\infty}\left\|\frac{1}{n}\sum_{i=1}^{n} x_{k(i)} - x\right\| = 0$,uniformly in the set of all strictly in - creasing sequences $(k(n))_n$ of natural numbers.

The equivalence of i)- iii) was proved by Grothendieck [4] and derives from an earlier criterion of weak compactness due to Dunford and Pettis.The condition iv),due to Pelczynski ,emphasizes the role of basic sequences in the problem under study.The equivalence of v)-vi) with i) is proved in [10] . H.P.Rosenthal has used Grothendieck 's results to express weak compactness (of a bounded subset of a space $L^1(\mu)$) in terms of relatively disjoint families. We shall need the following consequence of his theory

1.2 PROPOSITION. (H.P.Rosenthal [13]). Let E be a Banach space.

i) If $T \in L(c_0, E)$ is an operator such that $\inf \|Te_n\| > 0$, where $(e_n)_n$ denotes the natural basis of c_0, then there exists an infinite subset $M \subset N$ such that $T|c_0(M)$ is an isomorphism.

ii) If $T \in L(l^\infty, E)$ is an operator such that $T|c_0$ is an isomorphism then there exists an infinite subset $M \subset N$ such that $T|l^\infty(M)$ is an isomorphism.

2. ALMOST σ-COMPLETE BANACH LATTICES

The aim of this section is to discuss a certain generalization of the concept of (order) σ-completeness of a Banach lattice. The results which we obtain are similar with those proved by Dodds in [3] .

2.1 DEFINITION. A Banach lattice E is said to be almost σ-complete provided that for each order bounded sequence of pairwise disjoint positive elements x_n of E there exists an operator $T \in L(l^\infty, E)$ such that $Te_n = x_n$ for each $n \in N$. Here $(e_n)_n$ denotes the natural basis of c_0.

A sequence $(x_n)_n$ as in Definition 2.1 above is weakly summable and thus associated to an operator $T \in L(c_0, E)$. (Actually T is a lattice homomorphism from c_0 into a suitable E_x). If E is almost σ-complete then T extends to l^∞.

Clearly, every σ-complete Banach lattice is also almost σ-complete. Other examples are indicated below.

2.2 PROPOSITION. Let E be an almost σ-complete Banach lattice and let I be a closed ideal of E. Then E/I is also almost σ-complete. Particularly, the Banach lattice $C(\beta N \setminus N) = l^\infty/c_0$ is almost

σ -complete though it is not σ-complete.

Proof . Let $\pi : E \longrightarrow E/I$ the canonical surjection and let $(y_n)_n$ be a sequence of pairwise disjoint elements of E/I such that $0 < y_n \leq \pi(x)$ for a suitable $x \in E$, $x > 0$.Then by Lemma 2 in [1] there exists a sequence $(x_n)_n$ of pairwise disjoint elements of E such that $0 < x_n \leq x$ and $\pi(x_n) = y_n$ for each $n \in N$.The proof ends with an appeal to Definition 2.1 above .□

2.3 PROPOSITION. Assume the continuum axiom.Then every Banach lattice E having the interpolation property is almost σ-complete. (Recall that a Banach lattice E has the interpolation property provided that for any sequences $(x_n)_n$ and $(y_n)_n$ in E with $x_m \leq y_n$ for every $m,n \in N$, there exists an $x \in E$ such that $x_n \leq x \leq y_n$ for every n).

Proof.In fact, if E has the interpolation property then all the spaces E_x ($x \in E$, $x > 0$) have also the interpolation property .As noted in [15] ,a space C(S) has the interpolation property iff S is an F - space,i.e. disjoint open F_σ -subsets of S have disjoint closures. It remains to apply Lindenstrauss'result in [5] :By assuming the conti - nuum axiom it is true that each operator T from c_o into a space C(S), with S an F -space,extends to l^∞.□

2.4 PROPOSITION.Each complemented sublattice of an almost σ-complete Banach lattice is also almost σ-complete.

An example due to Bade (see [15] for details) shows that the inter - polation property does not pass to complemented sublattices .Conse - quently the almost σ -completeness does not coincide with the inter- polation property.

The main result of this section is the following extension of the Vi- tali -Hahn-Saks theorem in measure theory

2.5 THEOREM. Let E be an almost σ-complete Banach lattice, let $(x_n')_n \subset E'$ and suppose that $x'(x) = \lim_{n\to\infty} x_n'(x)$ exists for each $x \in E$. Then:

i) For each $0 \leq x \in E$, $\sup_n |x_n'(x_k)| \longrightarrow 0$ as $k \longrightarrow \infty$ for every disjoint sequence $(x_k)_k \subset [0,x]$;

ii) $x' \in E'$ and $x'(x) = \lim_{n\to\infty} x_n'(x)$ holds for all x in the ideal I_E generated by E in E''.

Proof. i) By Definition 2.1 above we may restrict ourselves to the case $E = l^\infty$, which was first treated by Grothendieck in [4], Theorem 9. The assertion ii) follows from i) and Theorem A in [2]. □

2.6 COROLLARY. Every almost σ-complete $C(S)$-space has the Grothendieck property, i.e. $x_n' \xrightarrow{w'} 0$ in $C(S)'$ implies $x_n' \xrightarrow{w} 0$. We do not know if the converse is true.

The proof follows from Theorem 1.1 ii) and Theorem 2.5 i) above. □

2.7 COROLLARY. Let E be an almost σ-complete Banach lattice, $B \subset E'$ a band and $P: E' \longrightarrow B$ the corresponding projection. If $(x_n')_n \subset E'$ and $x_n' \xrightarrow{w'} 0$ then $Px_n' \xrightarrow{w'} 0$. Consequently each band $B \subset E'$ is w' - sequentially complete.

Proof. Let us denote by Q the projection of I_E onto the carrier band of B in I_E. According to Theorem 27.12 in [9] it follows that $(Px')x = x'(Qx)$ for each $x \in I_E$, $x' \in E'$ and thus by Theorem 2.5ii) above we obtain that $(Px_n')x = x_n'(Qx) \longrightarrow 0$ for each $x \in I_E$. □

3. THE MAIN RESULTS

We start with the following

3.1 LEMMA. Let E be a Banach lattice, F a Banach space and $T \in L(E,F)$. Then the following assertions are equivalent:

i) T is of type $\mathcal{A}$;

ii) T maps every order interval of E into a relatively weakly compact subset of F ;

iii) T maps every order bounded sequence of pairwise disjoint elements into a norm convergent to 0 sequence ;

iv) T maps every order bounded sequence into a sequence with stable subsequences .

If in addition E is σ-complete then the conditions i)-iv) above are also equivalent with

v) There exists no sublattice X of E, lattice isomorphic to l^{∞}, such that $T|X$ is an isomorphism.

Proof. The condition ii) is equivalent with the fact that all compositions $T \circ i_x$ $(x \in E,\ x > 0)$ are weakly compact. Consequently the equivalence of the conditions i) - iv) follows from Theorem 1.1 above. Clearly, iii) implies v). We shall show that v) implies iii). For, let $(x_n)_n \subset [0,u]$ a sequence of pairwise disjoint elements of E and suppose that $\inf \|Tx_n\| > 0$. We consider the operator $S: l^{\infty} \longrightarrow E$ given by $S((a_n)_n) = (\sigma)\text{-} \sum a_n x_n$. Then $T \circ S$ verifies the assumptions of Proposition 1.2 ii) above and thus the restriction of T to a certain sublattice X of E, lattice isomorphic to l^{∞}, is an isomorphism, contradiction. □

3.2 THEOREM. Let E be an almost σ-complete Banach lattice, F a Banach space and $T \in L(E,F)$. Then the following assertions are equivalent :

i) T is of type A ;

ii) T" maps the ideal I_E (generated by E in E") into F ;

iii) T has the Pelczynski's property (u), i.e. for each weak Cauchy sequence $(x_n)_n$ in E there is a weakly summable sequence $(y_n)_n$ in $\overline{T(E)}$ such that $Tx_n - \sum_{k=1}^n y_k \xrightarrow{w} 0$;

iv) There is no subspace X of E, isomorphic to $C[0,1]$, such that $T|X$ is an isomorphism ;

v) There is no subspace X of E, isomorphic to l^∞, such that $T|X$ is an isomorphism .

Proof. i)$\Longrightarrow$ii). Let $Q:E \longrightarrow E"$ the canonical embedding and let $x \in E$, $x > 0$. Since i_x is interval preserving so is $(i_x)"$ (see [7]) and thus

$$T"[0,Qx] = T"[0,(i_x)"x] = (T \circ i_x)"[0,x] \quad .$$

If T is of type A then $T \circ i_x$ is weakly compact and thus $T"[0,Qx] \subset F$ for each $x \in E$, $x > 0$.

ii)$\Longrightarrow$iii). Without loss of generality we may assume that E is also separable. Then E (and also B_E, the band generated by E in E") has a weak order unit $u > 0$. Let $(x_n)_n$ be a weak Cauchy sequence in E. Since B_E is w'-sequentially complete, there exists a $z \in B_E$ such that $x_n \xrightarrow{w'} z$. See Corollary 2.7 above. Since B_E is an order complete vector lattice with a weak order unit, there exists a sequence $(z_n)_n$ of pairwise disjoint elements such that $|z_n| \leqslant nu$ and $z = (\sigma)\text{-} \sum z_n$. The sequence $(z_n)_n$ is w'-summable (with summ z) and contained in I_E. In fact, for each $x' \in E'$ we have $\sum |x'(z_n)| \leqslant \leqslant \sum |z_n|(|x'|) \leqslant |x'|(z)$ and $z(x') = \sum z_n(x')$. By ii) ,

$y_n = T''z_n \in F$ for each $n \in N$. The sequence $(y_n)_n$ being w'-summable in F'', it is also weakly summable in F. It is clear that

$y'(Tx_n - \sum_{k=1}^{n} y_k) \longrightarrow 0$ for each $y' \in F'$.

iii) $\Longrightarrow$ iv). In fact, it is well known that $C[0,1]$ contains the James' space J as a subspace and that 1_J fails the Pelczynski's property (u). On the other hand, the property (u) is hereditary. See [6] for details.

iv) $\Longrightarrow$ v). In fact, l^{∞} contains an isomorphic copy of $C[0,1]$.

v) $\Longrightarrow$ i). If T is not of type A then by Proposition 3.1 above there exist an $a > 0$ and a sequence $(x_n)_n \subset [0,x]$ of pairwise disjoint elements of E such that $\| Tx_n \| > a$. Since E is almost σ-complete, there exists an operator $S \in L(l^{\infty}, E)$ such that $Se_n = x_n$ for each $n \in N$. Here $(e_n)_n$ denotes the natural basis of c_0. Then Proposition 1.2 above yields a subspace X of E, isomorphic to l^{∞}, such that $T|X$ is an isomorphism.□

We pass now to the problem of characterizing the operators of type B. We shall need the following result concerning the reciprocal Dunford-Pettis property:

<u>3.3 LEMMA</u>. Let E be a Banach lattice which contains no lattice isomorph of l^1, F a Banach space and $T \in L(E,F)$. If T maps every weakly convergent sequence of pairwise disjoint elements into a norm convergent sequence then T is weakly compact.

See [11] for details.

<u>3.4 THEOREM</u>. Let E be a Banach lattice, F a Banach space and $T \in L(E,F)$. Then the following assertions are equivalent:

i) T is of type B ;

ii) $T \circ i_{x''}$ is weakly compact for every $x'' \in E''$, $x'' > 0$;

iii) If $(x_n)_n$ is a weakly summable sequence of pairwise disjoint positive elements of E then $\|Tx_n\| \longrightarrow 0$;

iv) If $(x_n)_n$ is a weakly summable sequence of positive elements of E then $\|Tx_n\| \longrightarrow 0$;

v) There exists no sublattice X of E, lattice isomorphic to c_0, such that $T|X$ is an isomorphism.

Proof. Clearly, i) $\Longrightarrow$ iv) $\Longrightarrow$ iii) $\Longleftrightarrow$ v).
iii) $\Longrightarrow$ ii). One applies Lemma 3.3 above. Each Banach lattice $E_{x''}$ is an AM-space and thus contains no lattice isomorph of l^1. Also, each norm bounded sequence of pairwise disjoint elements of $E_{x''}$ is equivalent to the natural basis of c_0 and thus it is weakly summable.
ii) $\Longrightarrow$ i). Each sequence $(x_n)_n$ E such that $0 < x_n\uparrow$ and $\|x_n\| \leq K$ can be viewed as a weak Cauchy sequence in a certain space $E_{x''}$. □

From Lemma 3.1 and Theorem 3.4 ii) it follows that each operator of type B is also of type A. A case when the converse is also true is indicated by the following:

3.5 PROPOSITION. Let E be a Banach lattice, F a Banach space and $T \in L(E', F)$ an operator of type A. Then T is also of type B.

Proof. Suppose that T is not of type B. Then by Theorem 3.4 there exists a weakly summable sequence of pairwise disjoint elements x_n of E such that $\|Tx_n\| \geq a > 0$. Then $X = \overline{\text{Span}}\,(x_n)_n$ is lattice isomorphic to c_0. Let $i: X \longrightarrow E'$ the canonical inclusion and let $P: E''' \longrightarrow E'$ the positive projection given by $(Px''')x = x'''(x)$ for all $x''' \in E'''$ and $x \in E$. By Proposition 1.2 ii) there exists an infinite subset $N_1 \subset N$ such that $T \circ P \circ i'' \mid l^{\infty}(N_1)$ is an isomorphism. Then $\tilde{X} =$

$\overline{\text{Span}}\ (x_n)_{n \in N_1}$ is lattice isomorphic to l^∞ and the restriction of T to $\tilde{X}$ is an isomorphism, in contradiction with the fact that T is an operator of type A. □

3.6 PROPOSITION. Let E be a Banach space, F a Banach lattice and $T \in L(E,F)$. Then the following statements are equivalent:

i) T' is of type B ;

ii) $j_{x'} \circ T$ is weakly compact for every $x' \in E'$, $x' > 0$;

iii) There is no complemented subspace X of E, isomorphic to l^1, such that $T(X)$ is complemented in F and $T|X$ is an isomorphism .

Every weakly compact operator is of type B. The converse is not generally true. A remarkable exception constitutes the case when E is a space $C(S)$. See [4] .

4. OPEN PROBLEMS.

The main problem which we leave open concerns the extensions properties of the operators of type B. An operator T defined on a Banach lattice E with values in a Banach space F is said to be of <u>strong type B</u> provided that T'' maps the band B_E , generated by E in E'', into F. Since B_E is the range of a (positive contractive) projection of E'' , such an operator extends to E''. Clearly, every operator of strong type B is also of type B.

<u>4.1 PROBLEM</u>. Does there exist an operator of type B which is not of strong type B ?